# Signals and Systems

Solutions to Homework Assignment #7

**Problem 1.**

a. The transformation from $X(s)$ to $Y(s)$ will be linear and causal because the components are linear and causal. Therefore, the system will be BIBO stable if and only if all of the closed-loop poles are in the left half plane. From Black's equation, the overall system response will be

$$H(s) = \frac{\frac{K}{s^2+s-2}}{1+\frac{K}{s^2+s-2}} = \frac{K}{s^2+s-2+K}.$$

Thus the closed-loop poles will be at

$$s = -\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 2 - K}.$$

The most positive root will be the least stable. Thus, the system will be stable if

$$-\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + 2 - K} < 0$$

$$\sqrt{\left(\frac{1}{2}\right)^2 + 2 - K} < \frac{1}{2}$$

$$\left(\frac{1}{2}\right)^2 + 2 - K < \left(\frac{1}{2}\right)^2$$

$$2 - K < 0$$

$$K > 2.$$

b. The closed-loop poles will be real-valued if

$$\left(\frac{1}{2}\right)^2 + 2 - K > 0$$

so that

$K < 2.25$.

**Problem 2.**

a. The transformation from $X(z)$ to $Y(z)$ will be linear and causal because the components are linear and causal. Therefore, the system will be BIBO stable if and only if all of the closed-loop poles are inside the unit circle. From Black's equation, the overall system response will be

$$H(z) = \frac{\frac{K}{z^2+z-2}}{1+\frac{K}{z^2+z-2}} = \frac{K}{z^2+z-2+K}.$$

Thus the closed-loop poles will be at

$$z = -\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 2 - K}.$$

The closed-loop poles will be on the real axis if $K < 2.25$ (from part a). Then

$$-1 < -\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 2 - K} < 1$$

$$-\frac{1}{2} < \pm\sqrt{\left(\frac{1}{2}\right)^2 + 2 - K} < \frac{3}{2}$$

so that

$$\sqrt{\left(\frac{1}{2}\right)^2 + 2 - K} < \frac{3}{2}$$

and

$$-\sqrt{\left(\frac{1}{2}\right)^2 + 2 - K} > -\frac{1}{2}.$$

The first will be true if $K > 0$ and the second will be true if $K > 2$. Together, these imply that $K$ must be bigger than 2. The closed-loop poles will be complex if $K > 2.25$. For this case

$$\left|-\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 2 - K}\right| < 1.$$

Thus

$$\left(\frac{1}{2}\right)^2 + K - 2 - \left(\frac{1}{2}\right)^2 < 1$$

so that

$$K < 3.$$

Combining the results for real and complex cases,

$$2 < K < 3.$$

b. The closed-loop poles will be real-valued if

$$\left(\frac{1}{2}\right)^2 + 2 - K > 0$$

so that

$$K < 2.25$$

(same as in 1b).

**Problem 3.**

a. From Black's equation, the overall system response will be

$$H(s)=\frac{\frac{s^2-2s+2}{(s+1)^2}}{1+K\frac{s^2-2s+2}{(s+1)^2}}=\frac{s^2-2s+2}{s^2+2s+1+Ks^2-2Ks+2K}=\frac{s^2-2s+2}{(1+K)s^2+2(1-K)s+1+2K}.$$

The closed-loop poles will be at

$$s=\frac{2(K-1)\pm\sqrt{4(1-K)^2-4(1+K)(1+2K)}}{2(1+K)}=\frac{K-1}{K+1}\pm\frac{\sqrt{-K(K+5)}}{(K+1)}$$

b. When $K = 0$, the closed-loop poles are at the same positions as the open-loop poles: there is a double pole at $s = -1$. This condition is therefore stable. As $K$ is increased, $-K(K + 5)$ becomes negative, which makes the closed-loop poles become complex. These complex poles will remain stable so long as the real part $(K - 1)/(K + 1)$ is negative, i.e., for $0 < K < 1$. As $K$ gets very large, the closed-loop poles converge toward $s = 1 \pm j$, and the system is unstable. When $K$ is slightly negative, $-K(K + 5)$ is positive. Thus the double pole that results for $K = 0$ begins to split: giving rise to a pole that is at $s < -1$ and a pole at $s > -1$. This pair will remain stable until $K = -1/2$ at which point the right most pole is at $s = 0$. In total, the system is stable for $-1/2 < K < 1$.

Alternatively, we can use Routh-Hurwitz criteria as covered in tutorial. One thing to note in this case, however, is that we simply need to have all the coefficients in the denominator of $H(s)$ to be of the same sign. Thus, when we consider the case when all the coefficients are positive, we have

$$\begin{cases}1+K & >0\\ 2(1-K) & >0\\ 1+2K & >0\end{cases}\quad\Rightarrow\quad\begin{cases}K & >-1\\ K & <1\\ K & >-\frac{1}{2}\end{cases}\quad\Rightarrow\quad -\frac{1}{2}<K<1,$$

which is consistent with the region obtained above. On the other hand, if we consider the case when all the coefficients are negative, we have

$$\begin{cases}1+K & <0\\ 2(1-K) & <0\\ 1+2K & <0\end{cases}\quad\Rightarrow\quad\begin{cases}K & <-1\\ K & >1\\ K & <-\frac{1}{2}\end{cases}\quad\Rightarrow\quad \text{no such } K!$$

c. In order to have persistent oscillation, we want to have the closed loop system with no damping, i.e., the coefficient of s is zero. Thus, $1 - K = 0 \Rightarrow K = 1$. Then, $H(s)$ becomes

$$H(s)=\frac{s^2-2s+2}{2s^2+3}=\frac{1}{2}\cdot\frac{s^2-2s+2}{s^2+\frac{3}{2}}.$$

Thus, the oscillation frequency is $\sqrt{\frac{3}{2}}$.

Alternatively, we want to have no real parts in the closed-loop poles, thus,

$$\frac{K-1}{K+1}=0\quad\Rightarrow K=1.$$

**Problem 4.** Let *W*(*s*) represent the input to the *A*(*s*) system and let *Z*(*s*) represent the output of that system. Then, using Black's equation, we can write that

$$W(s) = \frac{1}{1 + A(s)C(s)} X(s)$$

and

$$Z(s) = \frac{A(s)}{1 + A(s)C(s)} X(s).$$

Then

$$Y(s) = B(s)W(s) + Z(s) = B(s)\frac{1}{1 + A(s)C(s)} X(s) + \frac{A(s)}{1 + A(s)C(s)} X(s)$$

$$= \frac{A(s) + B(s)}{1 + A(s)C(s)} X(s).$$

Thus

$$H(s) = \frac{A(s) + B(s)}{1 + A(s)C(s)}.$$