Approximately Universal Codes for Slow Fading Channels

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A Wireless Channel

- Slow Fading channel

$$y[m] = h\,x[m] + w[m]$$

- Multiplicative noise h fixed over time scale of communication

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- Reliable communication:

  Fundamental tension between data rate R and error probability $\mathbb{P}_e$

- Simple observation: arbitrarily reliable communication not possible at any data rate

  Error Probability $\mathbb{P}_e > 0$

Rate and Probability of Error

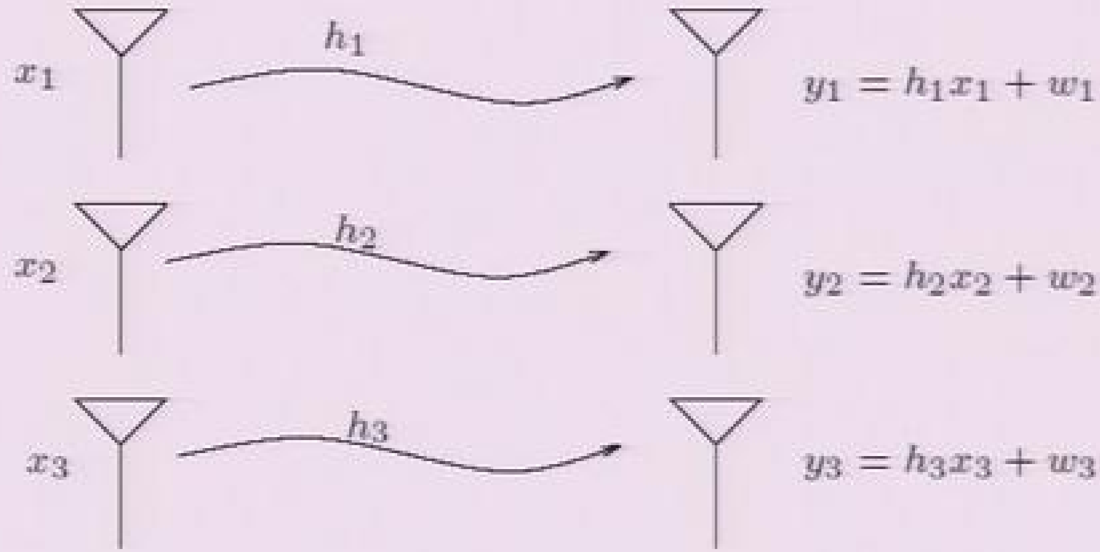
- Tradeoff between rate R, and probability of error $\mathbb{P}_e$
- Outage: Given a rate and SNR:

$$\begin{aligned} \mathbb{P}_{\text{out}} &= \mathbb{P}\left\{h \mid I(x;y|\,h) \;<\; R\right\} \\ &= \mathbb{P}\left[\log\left(1+|h|^2\mathsf{SNR}\right) < R\right]. \end{aligned}$$

A Summary

- ◇ A narrowband slow fading channel is ordered.
- ◇ An AWGN channel capacity-achieving code works here as well.
- ◇ Several channel models are not ordered:
  - a parallel channel
  - a MIMO channel

Parallel Fading Channel



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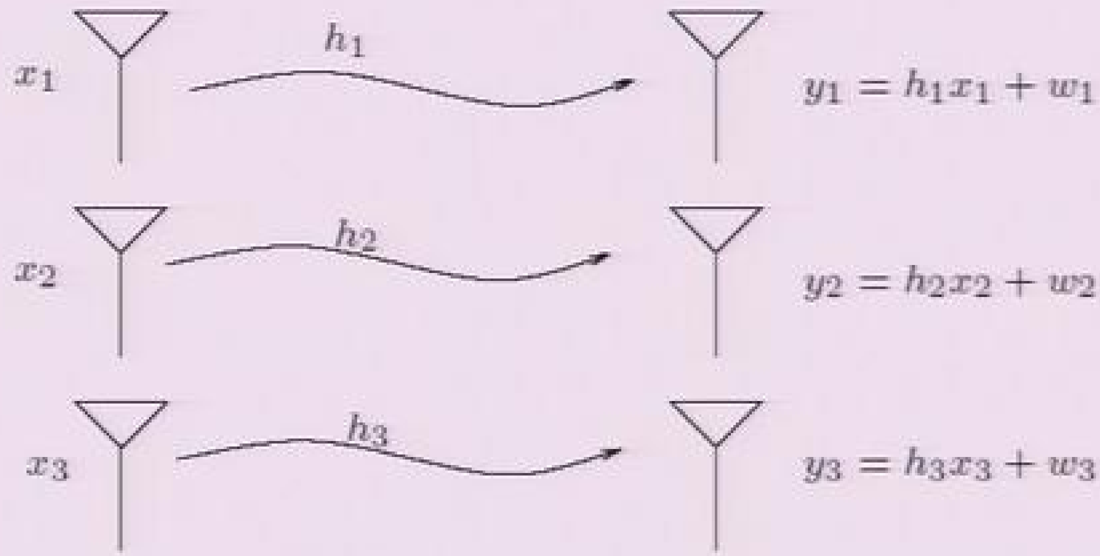

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Communication Over Slow Fading Channel

- L parallel sub-channels
- Slow fading: $h_1, \ldots, h_L$ random, but fixed over time
- Correlated fading: $h_1, \ldots, h_L$ jointly distributed
- Coherent communication: $h_1, \ldots, h_L$ known to the receiver

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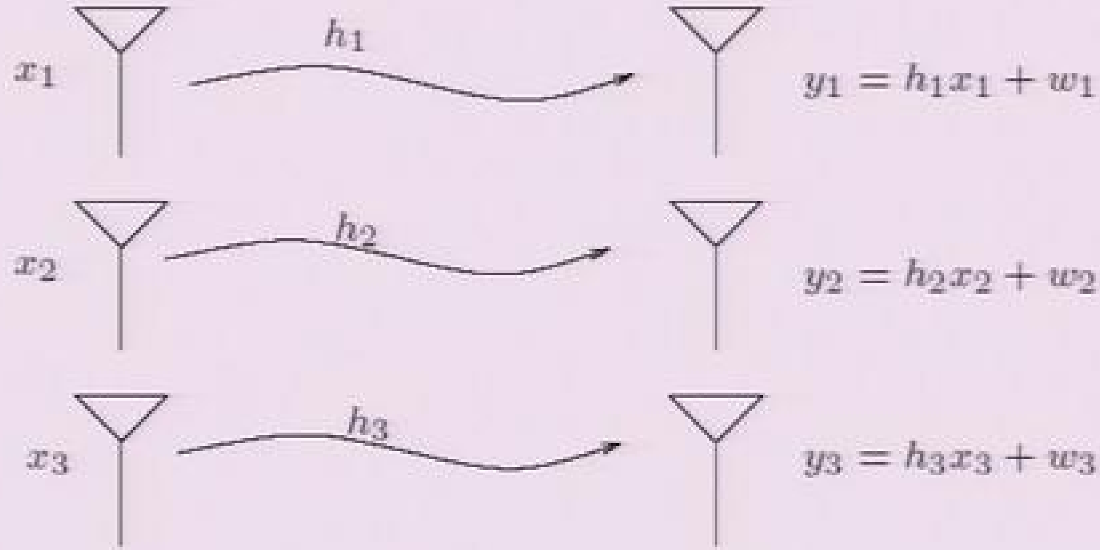

- Time Diversity: coding over time
- Frequency Diversity: coding over OFDM symbols
- Antenna Diversity: coding for MIMO channel: D-BLAST

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- Focus in this talk:

  Short block-length communication at high SNR

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$$\mathbb{P}_{\text{out}} := \min_{\mathbb{P}_{\boldsymbol{x}}} \mathbb{P}(\{\, \mathbf{h} \mid I(\boldsymbol{x};\boldsymbol{y}|\,\mathbf{h}) < R \,\})$$

- Compound channel result:

Universal code achieves reliable communication for all channels not in outage

- $\mathbb{P}_e = \mathbb{P}_{\text{out}}$ with universal codes.

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- Understand universal codes at high SNR
  - code design criteria
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R and $\mathbb{P}_e$: a Coarser Scaling

- Coarser formulation (ZT03):
  - Rate $= r \log(\mathsf{SNR})$
  - Probability of error $= \frac{1}{\mathsf{SNR}^d}$
- Given r, find maximal $d = d^*(r)$
- Allows us to focus on the fading coefficient h rather than the combination of the fading coefficient and the additive noise.

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Characterization of Channels in Outage

- Outage: Input distribution can be taken as i.i.d. Gaussian

$$\text{outage} = \left\{ \mathbf{h} \mid \sum_{i=1}^{L} \log\left(1 + |h_i|^2 \mathsf{SNR}\right) < r \log(\mathsf{SNR}) \right\}$$

- Outage condition independent of distribution on $\mathbf{h}$

- Outage curve: $\mathbb{P}(\text{outage}) = \mathsf{SNR}^{-d_{\text{out}}(r)}$

- $\mathbb{P}(\text{outage}) \leq \mathbb{P}_e \Rightarrow d^*(r) \leq d_{\text{out}}(r)$

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- Main Result:
  - Space only codes universally achieve the outage curve for the parallel channel
    - engineering value: code is robust to channel modeling errors
  - Simple deterministic construction of permutation codes
- Use universal parallel channel codes as constituents of universal MIMO channel codes

Smart Union Bound

- $\mathbb{P}(\text{outage}) \leq \mathbb{P}_e \leq \mathbb{P}(\text{outage}) + \mathbb{P}(\text{error}|\text{no-outage})$

- We want the second term to decay exponentially in SNR
  - Look at the union bound
  - Each pairwise error should decay exponentially in SNR

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- Let $\mathbf{d} = \boldsymbol{x}(1) - \boldsymbol{x}(2)$ be the difference codeword (space-only)

$$\mathbb{P}_\text{e}\left(\boldsymbol{x}(1) \rightarrow \boldsymbol{x}(2) \mid \mathbf{h}\right) \;\le\; \exp\left(-\sum_{i=1}^{L} |h_i|^2 |d_i|^2\right)$$

Code Construction Criterion

- Worst case analysis over all realizations not in outage:

$$\min_{\mathbf{h} \notin \text{outage}} \left[\sum_{i=1}^{L} |h_i|^2 |d_i|^2 \right]$$

$$\mathbf{h} : \sum_{i=1}^{L} \log\left(1 + \mathsf{SNR}|h_i|^2\right) > R$$

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- Outage condition independent of the distribution of $\mathbf{h}$
  - Code construction is universal
  - Viewpoint taken by (Wes95, KW03)
- Contrast with the traditional analysis: average over the channel statistics

Code Construction Criterion

- Worst case analysis over all realizations not in outage:

$$\min_{\mathbf{h} \notin \text{outage}} \left[\sum_{i=1}^{L} |h_i|^2 |d_i|^2 \right] > 1$$

$$\mathbf{h} : \sum_{i=1}^{L} \log\left(1 + \mathsf{SNR}|h_i|^2\right) > R$$

- Related to the water-pouring problem
  - Constraint function and objective function reversed

$$|h_i|^2 = \left(\frac{1}{\lambda} - |d_i|^2\right)^+$$

Code Construction Criterion

- Turns out to be the product distance:

$$|d_1|^2|d_2|^2 \cdots |d_L|^2 \geq \frac{1}{2^R}$$

- Also a necessary condition.
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Revisit Code Design Criterion

$$|d_1|^2|d_2|^2 \cdots |d_L|^2 \geq \frac{1}{2^R}$$

- Nonzero product distance
  - Need each sub-channel to have all the information
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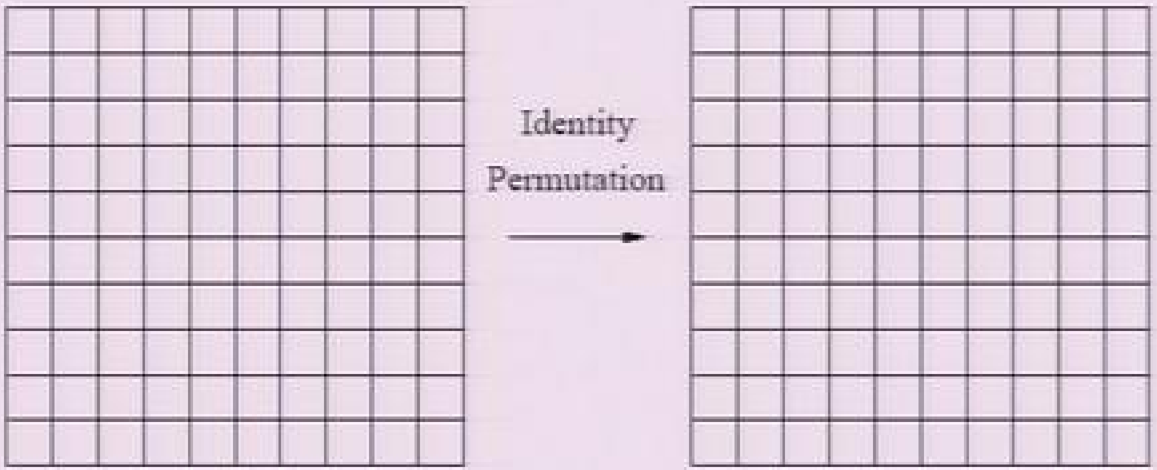
Implications on Code Structure

- Nonzero product distance
  - Need each sub-channel to have all the information
  - So, alphabet size 2^R
  - Can take it to be a QAM ($\mathbb{Q}$) with 2^R points
- Mapping across sub-channels
  - Each point in the QAM for any sub-channel should represent the entire codeword
  - So, code is $L-1$ permutations of $\mathbb{Q}$

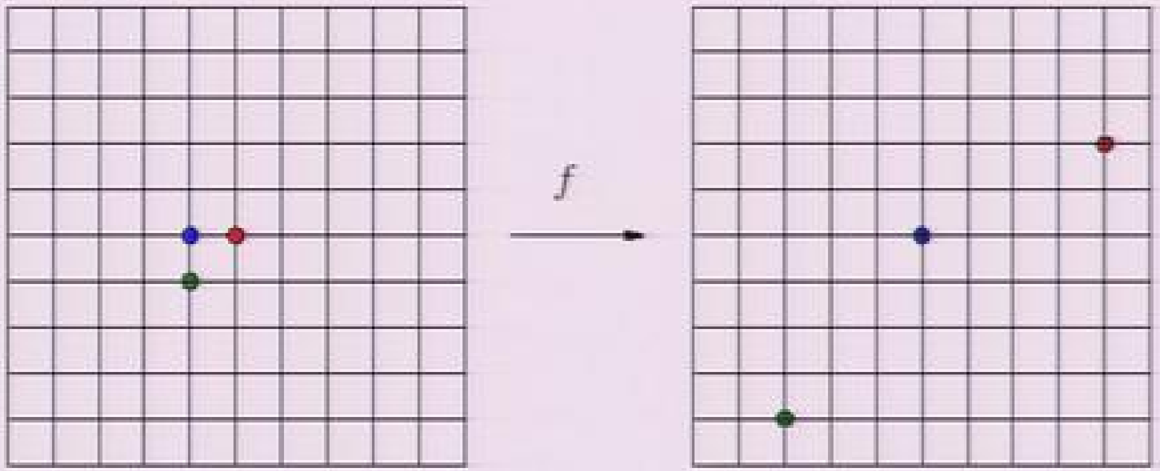
Repetition Coding

- the identity permutation



- $L = 2$, product distance $= \frac{1}{2^{2R}}$
- We want: product distance $> \frac{1}{2^R}$

Key Property of Permutations



- Two adjacent points should be mapped as far apart as possible

Random Permutation Codes

- Look at the ensemble of all permutation codes
  - huge number of permutations: $(2^R!)^{L-1}$
  - average product distance under appropriate measure
- There exists a permutation which satisfies the product distance
- Conclusion: A permutation code achieves universally the outage curve

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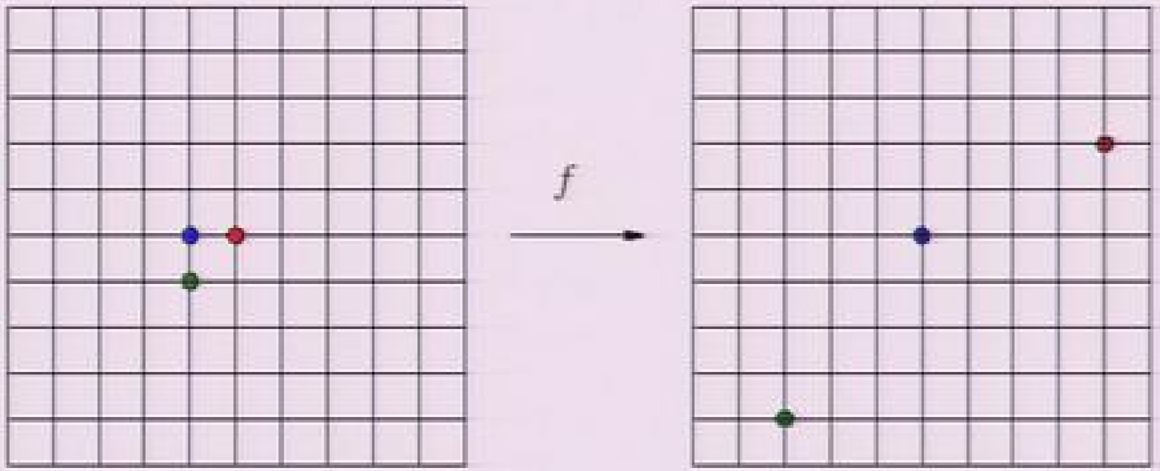

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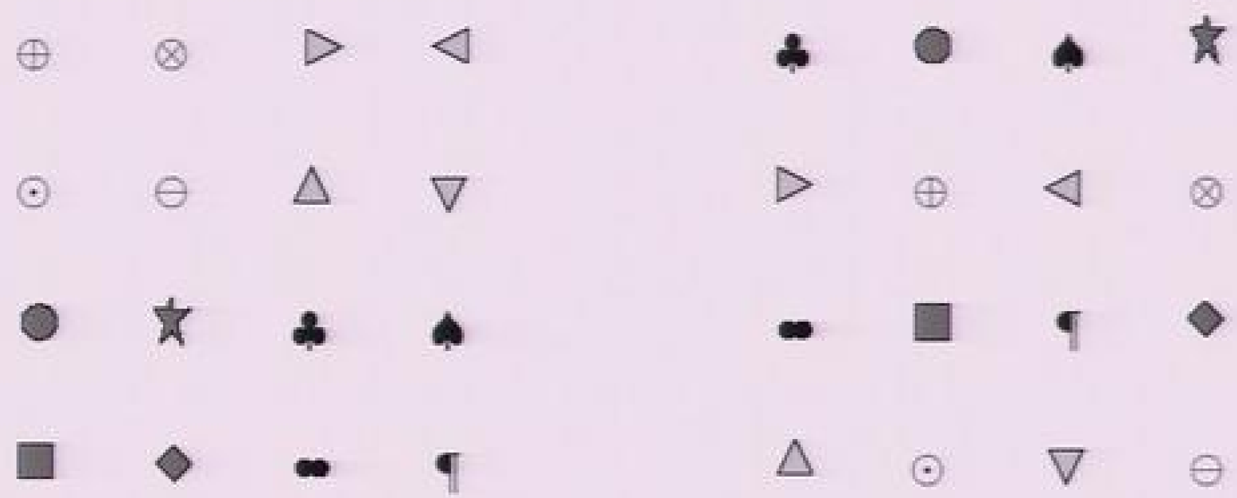
An Example

- 16-point Permutation Code



2 Sub-Channels

- Transmit $q \in \mathbb{Q}$ and $f(q) \in \mathbb{Q}$ over the two sub-channels

$$
\begin{aligned}
q &= a + ib, \qquad a, b \;\; \text{integers} \\
y_1 &= h_1\,(a + ib) \,+\, w_1 \\
y_2 &= h_2\,f(a + ib) \,+\, w_2
\end{aligned}
$$

Effect of the Fading Channel

- Consider binary representation of integers a and b
  - require $n = \frac{R}{2}$ bits
- Additive Gaussian noise very likely to move within neighboring integers

2-sub Channels

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- Look at permutations ($\tilde{f}$) of real and imaginary values

$$
\begin{aligned}
\tilde{y}_1 &= |h_1|\,(a + ib) \,+\, \tilde{w}_1 \\
\tilde{y}_2 &= |h_2|\,(\tilde{f}(a) + i\tilde{f}(b)) \,+\, \tilde{w}_2
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Effect of the Fading Channel

- Consider binary representation of integers a and b
  - require $n = \frac{R}{2}$ bits
- Additive Gaussian noise very likely to move to neighboring integers
- Effect of the multiplicative channel: distort the LSBs

$$|h_1| \approx 2^{-k_1} \implies k_1 \text{ LSBs of } a \text{ and } b \text{ lost}$$
$$|h_2| \approx 2^{-k_2} \implies k_2 \text{ LSBs } f(a) \text{ and } f(b) \text{ lost}$$

- No outage condition:

$$|h_1||h_2| > \frac{2^{R/2}}{\mathsf{SNR}} \implies k_1 + k_2 \leq n$$

Bit Reversal Permutation

- Bit reversal: $\tilde{f}(a)$ is bit reversal of a
- Encoding:
  - Same complexity as encoding a QAM
- Decoding:
  - Use first sub-channel to determine MSBs of a and b
  - Use second sub-channel to determine LSBs of a and b
  - No outage condition means you recover all the bits

Bit Reversals

- Bit-reversal permutation code is approximately universal
- Bit-reversal with alternate bits flipped is even better:
- Theorem: For every pair of integers a_1, a_2 between 0 and $2^R - 1$,

$$|a_1 - a_2| \cdot |BR(a_1) - BR(a_2)| \geq \frac{2^R}{8}.$$

3 Sub-Channels

- First sub-channel: identity permutation
- Second sub-channel: reverse the bits
- Third sub-channel: Pass the bits through the linear transformation

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- More generally,

$$A_{2n} = A_n \otimes \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

A Combinatorial Open Problem

- Combinatorial Design: Matrices $A_n^{(1)}, \ldots, A_n^{(L)}$ with elements from finite field $\mathbb{F}_q$ of size q such that

  the first k_j rows of A_j, with $j = 1 \ldots L$, should span $\mathbb{F}_q$

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- Open Questions:
  - What is the smallest field size q for which design exists?
  - Canonical representation for the designs?

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- Related literature:
  - Random matrices $A_n^{(1)}, \ldots, A_n^{(L)}$
  - MDS (maximum distance separable) codes – classical material in coding theory: Reed-Solomon codes

A Conjecture for $L = 4$

- Need at least ternary representation of integers: so $q \geq 3$
- Second sub-channel: reverse the ternary digits
- Third sub-channel: $A_{3n}^{(3)} = A_n^{(3)} \otimes \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
- Fourth sub-channel: $A_{3n}^{(4)} = A_n^{(4)} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

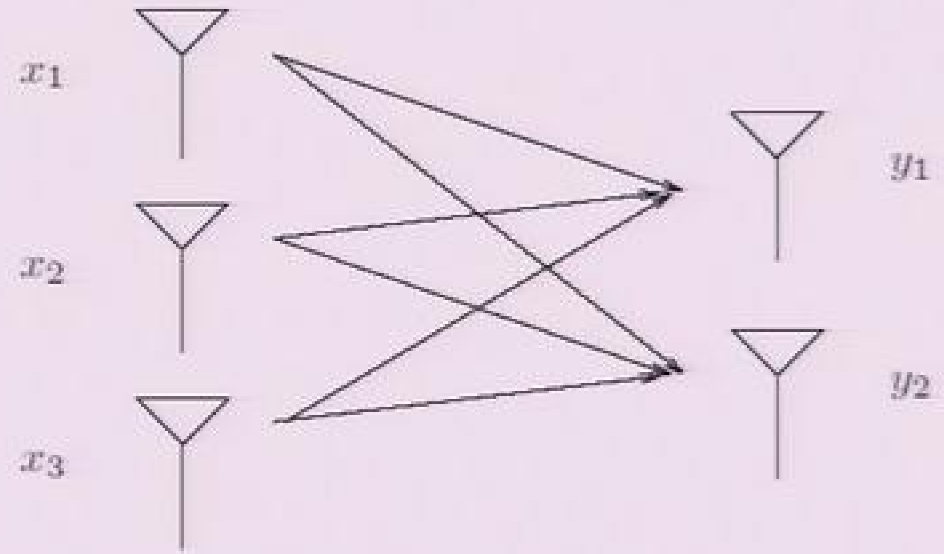
Approximately Universal Codes

- A code is approximately universally if it is tradeoff optimal for every channel distribution
- Universal code design criterion

$$\lambda_1 \lambda_2 \ldots \lambda_{\min(n_r, n_t)} \geq \frac{1}{2^R}$$

- $\lambda_1 \leq \ldots \leq \lambda_{n_t}$ are eigenvalues of $\mathbf{D}\mathbf{D}^\dagger$, $\mathbf{D} := \mathbf{X}(1) - \mathbf{X}(2)$
- Need this criterion for every pair of codewords

Fading MIMO Channel



- $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w}$
- Entries of $\boldsymbol{H}$ have a joint distribution.
  - slow fading, coherent

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- $\lambda_1 \leq \ldots \leq \lambda_{n_t}$ are eigenvalues of $\mathbf{D}\mathbf{D}^\dagger$, $\mathbf{D} := \mathbf{X}(1) - \mathbf{X}(2)$
- Need this criterion for every pair of codewords

Engineering Appeal

- Approximately Universal code over an $n \times n$ channel is also approximately universal over every

$$n \times n_r \quad \text{channel}$$

Approximately Universal Codes

- A code is approximately universally if it is tradeoff optimal for every channel distribution
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$$n \times n_r \quad \text{channel}$$

- Can use approximately universal code at the base station of downlink to transmit common information
  - code performance best possible for any number of receive antennas at the mobile users
  - code performance best possible with respect to the underlying statistical model of the fading channel

A Contrast

- Consider a MISO channel: $n_r = 1$
- Traditional design criterion for i.i.d. Rayleigh fading (TSC98)

$$\text{maximize} \quad \lambda_1 \lambda_2 ... \lambda_{n_t} \quad \text{(determinant)}$$

- Universal design criterion

$$\text{maximize} \quad \lambda_1 \quad \text{(smallest singular value)}$$

  - have to protect against the worst case channel

Universal Codes

- Need rotational invariance

$$XX^* = \text{SNR} \quad I$$

  - full rate orthogonal designs are universal

Universal Codes

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- Nearly rotational invariant: codes based on number theory
  - Tilted QAM code (YW03, DV03), Golden code (BRV04), codes based on cyclic division algebra (Eli04)
  - Universally tradeoff optimal, but hard to decode

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full rate orthogonal designs are universal

- Main Result:
  - Universal tradeoff optimal designs based on parallel channel codes
    - engineering value: code is robust to channel modeling errors
    - Simple encoding and decoding of permutation codes

Restricted Universality

- Setting:
  - coherent communication over short block length at high SNR
  - universal tradeoff performance over a restricted class of channels

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MISO Channel Revisited

$$\mathbf{y}[m] = \mathbf{h}^* \mathbf{x}[m] + \mathbf{w}[m]$$

- Suppose $h_1, \ldots, h_{n_t}$ are i.i.d.
  - no antenna is particularly vulnerable
- Universality Result:

  It is tradeoff optimal to use one transmit antenna at a time

- Converts MISO channel into a parallel channel
  - can use the simple universal permutation codes

Restricted Class of Channels

$$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^*$$

- $\mathbf{V}^*$ is the Haar measure on $SU(n_t)$ and $\mathbf{\Lambda}$ independent of $\mathbf{V}$
- each transmit direction is equally likely
- i.i.d. Rayleigh fading belongs to this class

D-BLAST Architecture

- D-BLAST scheme for 2 transmit antennas:

$$X = \begin{bmatrix} 0 & a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 & 0 \end{bmatrix}$$

- Successive cancellation of streams
  - converts MIMO channel to a parallel channel
  - can use permutation codes
- Almost universally tradeoff optimal
  - initialization overhead

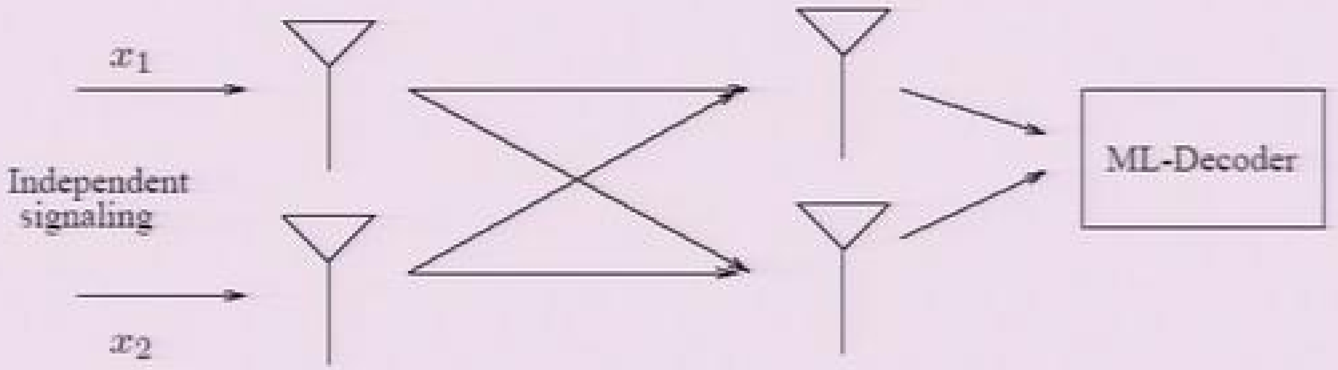
D-BLAST with ML decoding

$$X = \begin{bmatrix} 0 & a_2 & b_2 \\ a_1 & b_1 & 0 \end{bmatrix}$$

- Consider joint ML decoding of both streams
- Claim: Universal tradeoff performance same as that with successive cancellation
- Main result:

Universal tradeoff performance over the restricted class of channels

V-Blast with ML decoding



◇ Send QAM independently across antennas (space only code)

Reprise

- Considered universal tradeoff optimality
- Main results:
- Simple permutation codes for the parallel channel
- For a restricted class of MIMO channels

  Universal tradeoff optimality of D-BLAST and V-BLAST
- Reference:

  Chapter 9 of *Fundamentals of Wireless Communication*,
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Reprise

- Approximately universal codes

- Main results:
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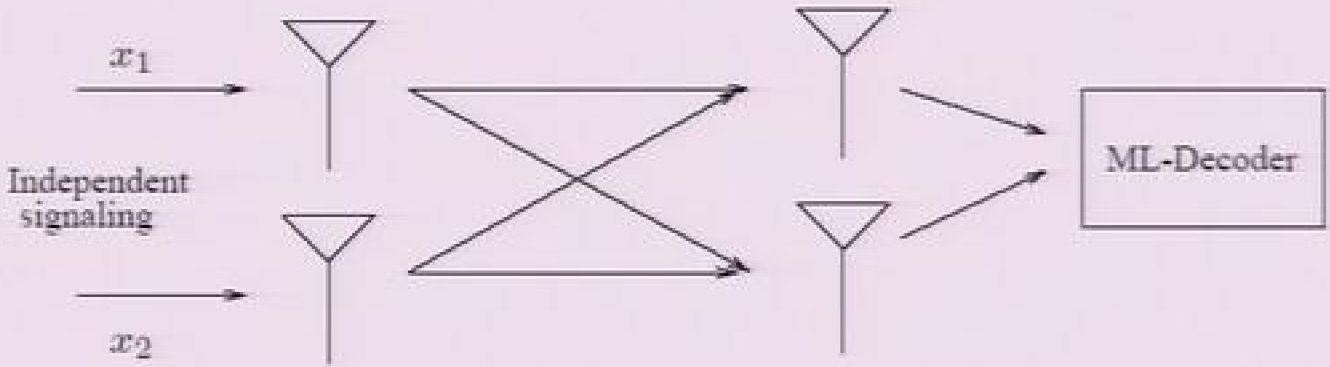

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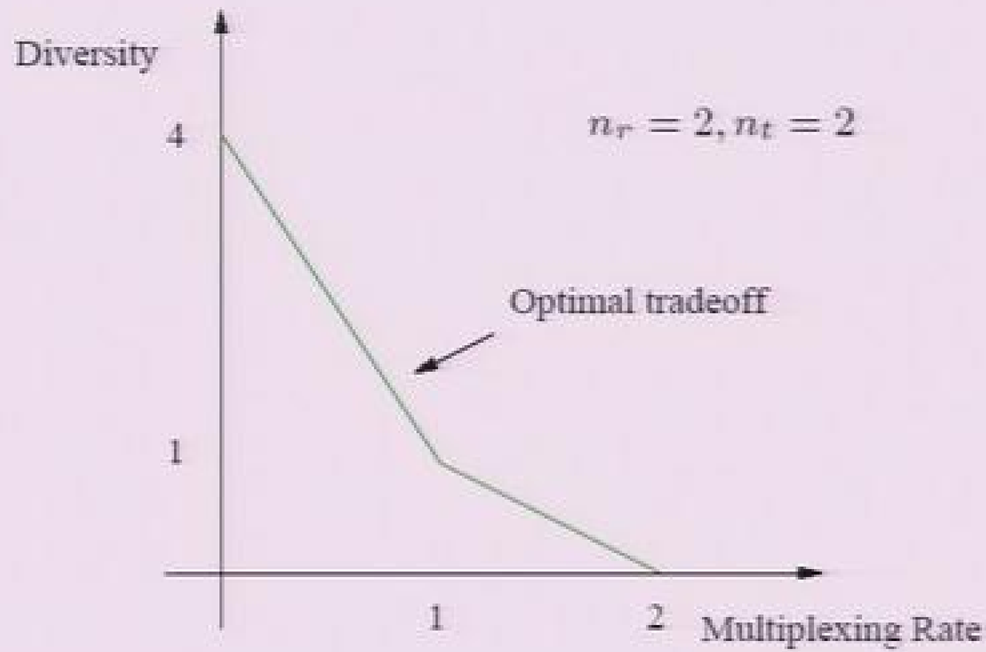
2×2 i.i.d. Rayleigh channel



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A Combinatorial Open Problem

- Combinatorial Design: Matrices $A_n^{(1)}, \dots, A_n^{(L)}$ with elements from finite field $\mathbb{F}_q$ of size q such that

  the first k_j rows of A_j, with $j = 1 \dots L$, should span $\mathbb{F}_q$

- Related literature:
  - Random matrices $A_n^{(1)}, \dots, A_n^{(L)}$
  - MDS (maximum distance separable) codes – classical material in coding theory: Reed-Solomon codes

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3 Sub-Channels

- First sub-channel: identity permutation
- Second sub-channel: reverse the bits
- Third sub-channel: Pass the bits through the linear transformation

$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- Open Questions:
  - What is the smallest field size q for which design exists?
  - Canonical representation for the designs?